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1994 J. Phys. A: Math. Gen. 27 6433

(http://iopscience.iop.org/0305-4470/27/19/016)

J. Phys. A: Math. Gen. 27 (1994) 6433–6442. Printed in the UK

# Unified approach to the analogues of single-photon and multiphoton coherent states for generalized bosonic oscillators

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Received 23 May 1994

**Abstract.** A large class of bosonic coherent states known in literature have been constructed in a unified way by Shanta *et al*. It is shown that this method can be easily extended to generalized bosonic-oscillator systems.

## 1. Introduction

An aspect of the theory of quantum groups is the notion of deformed, or generalized, bosonic oscillators. These generalized bosonic oscillator (GBO) systems have the potential to be useful in describing physical phenomena in which anharmonicity effects are sufficiently large. This motivates the study of GBO coherent states which would naturally take the place of the usual bosonic coherent states in applications. As has been often recognized by several authors in the literature, any GBO algebra can be presented in the form

$$[N, a^\dagger] = a^\dagger \qquad [N, a] = -a \qquad a^\dagger a = \phi(N) \qquad aa^\dagger = \phi(N+1) \tag{1.1}$$

where $a$, $a^\dagger$ and $N$ are, the annihilation, creation and excitation number operators, respectively, and the real non-negative function $\phi(N)$ ($\phi(n) \geqslant 0\ \forall\ n \geqslant 0$) characterizes the given system. For the usual boson $\phi(N) = N$, the $q$-oscillators [1–13], the $(p, q)$-oscillator [14–16] and the various other single-mode deformed bosons [17–21] and parabosons [22–25] correspond to special cases of algebra (1.1), each characterized by a $\phi(N)$. We assume that system (1.1) has a unique vacuum state $|0\rangle$ such that $a|0\rangle = 0$, $N|0\rangle = 0$ and the spectrum of $N$ is taken to be $\{0, 1, 2, \ldots\}$. Then, $\phi(N)|0\rangle = \phi(0)|0\rangle = 0$ and, throughout this paper, we shall assume that $\phi(N) > 0\ \forall\ n > 0$. For any GBO, the specific commutation relation connecting $a$, $a^\dagger$ and $N$ is derived from the recursion relation for $\phi(N)$. Following the construction [2] of the coherent state for the $q$-oscillator with $aa^\dagger - qa^\dagger a = q^{-N}$, there have been many studies on the $q$-coherent states [26–34]. The main purpose of this article is to construct the analogues of the single-photon and multiphoton coherent states for the GBO system (1.1) in a unified way. We do this by a straightforward application of the technique that has been developed recently [35] for constructing a large class of bosonic coherent states known in literature in a unified way. To this end, we use the fact that for any GBO (1.1), one can define [10, 11, 22] a pair of operators $(A, A^\dagger)$ such that

$$[A, a^\dagger] = 1 \qquad [a, A^\dagger] = 1 \qquad a^\dagger A = A^\dagger a = N. \tag{1.2}$$

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0305-4470/94/196433+10$19.50  

Explicitly,

$$A = \frac{N+1}{\phi(N+1)} a \qquad A^\dagger = a^\dagger \frac{N+1}{\phi(N+1)}. \tag{1.3}$$

As should be expected, the pairs $(a, A^\dagger)$ and $(A, a^\dagger)$ in the bosonic realization are related to the bosonic operators $(b, b^\dagger)$ through a similarity transformation. Hence, the formulae for the multiphoton coherent states are readily translated into their analogues for the GBO (1.1).

## 2. Multiphoton coherent states: A résumé

Let us recall briefly the recently developed general method for the construction of multiphoton coherent states [35]. Let $F$ be an operator consisting of products of annihilation operators ($b$) and number operators ($N$). For each mode $[b, b^\dagger] = 1$, $[N, b] = -b$, $[N, b^\dagger] = b^\dagger$ and $N = b^\dagger b$. Let $\{|\nu\rangle_i | i = 0, 1, 2, \ldots\}$ be the set of states which are annihilated by $F$: $F|\nu\rangle_i = 0$, $i = 0, 1, 2, \ldots$. If $G_i^\dagger$ is an operator satisfying the relation

$$[F, G_i^\dagger] = 1 \tag{2.1}$$

in the space $\mathcal{S}_i$ of states spanned by the set $S_i = \{F^{\dagger n}|\nu\rangle_i | n = 1, 2, \ldots\}$, then the states

$$|\alpha\rangle_i \sim \exp(\alpha G_i^\dagger)|\nu\rangle_i \qquad i = 1, 2, \ldots \tag{2.2}$$

are seen to be distinct eigenstates of $F$ with eigenvalue $\alpha$, provided they have finite norms. The operator $G_i = (G_i^\dagger)^\dagger$ consists precisely of the annihilation operators present in $F$ and, hence, $G_i|\nu\rangle_i = 0$, $i = 1, 2, \ldots$. Consequently, one has

$$[G_i, F^\dagger] = 1 \tag{2.3}$$

in the space $\mathcal{S}_i$ and the states

$$|\alpha\rangle_i' \sim \exp(\alpha F^\dagger)|\nu\rangle_i \qquad i = 1, 2, \ldots$$

are, respectively, the eigenstates of $G_i$ with eigenvalue $\alpha$.

For example, let $F = b^2$. We have $|\nu\rangle_1 = |0\rangle$, $|\nu\rangle_2 = |1\rangle$. Then, with $\mathcal{S}_1$ and $\mathcal{S}_2$ denoting the spaces spanned by the sets of states $S_1 = \{|2n\rangle | n = 0, 1, 2, \ldots\}$ and $S_2 = \{|2n+1\rangle | n = 0, 1, 2, \ldots\}$

$$G_1^\dagger = \frac{1}{2(N-1)} b^{\dagger 2} \qquad G_2^\dagger = \frac{1}{2N} b^{\dagger 2} \tag{2.5}$$

are such that

$$\begin{aligned} [F, G_1^\dagger]|\psi\rangle_1 &= |\psi\rangle_1 \qquad \forall\ |\psi\rangle_1 \in \mathcal{S}_1 \\ [F, G_2^\dagger]|\psi\rangle_2 &= |\psi\rangle_2 \qquad \forall\ |\psi\rangle_2 \in \mathcal{S}_2. \end{aligned} \tag{2.6}$$

Hence,

$$|\alpha\rangle_1 \sim \exp\left(\frac{\alpha}{2(N-1)} b^{\dagger 2}\right)|0\rangle \qquad |\alpha\rangle_2 \sim \exp\left(\frac{\alpha}{2N} b^{\dagger 2}\right)|1\rangle \tag{2.7}$$

are two distinct eigenstates of $b^2$ corresponding to the eigenvalue $\alpha$ (for details of the procedure for constructing the operators $\{G_i^\dagger\}$ for a given $F$, see [35]) and

$$|\alpha\rangle_1' \sim \exp(\alpha b^{\dagger 2})|0\rangle \qquad |\alpha\rangle_2' \sim \exp(\alpha b^{\dagger 2})|1\rangle \tag{2.8}$$

are, respectively, the eigenstates of $G_1 = \frac{1}{2(N+1)}b^2$ and $G_2 = \frac{1}{2(N+2)}b^2$ corresponding to the eigenvalue $\alpha$.

As another example, let $F = b_1 b_2$ where $(b_1, b_1^\dagger)$ and $(b_2, b_2^\dagger)$ are two commuting pairs of bosonic operators. Now, the distinct $\{|\nu\rangle_i\}$ and the appropriate $S_i$ are

$$\begin{aligned}
&|\nu\rangle_{m1} = |m, 0\rangle \qquad && S_{m1} = \{|m+n, n\rangle | n = 0, 1, 2, \ldots, m > 0\} \\
&|\nu\rangle_{m2} = |0, m\rangle \qquad && S_{m2} = \{|n, m+n\rangle | n = 0, 1, 2, \ldots, m > 0\} \\
&|\nu\rangle_3 = |0, 0\rangle \qquad && S_3 = \{|n, n\rangle | n = 0, 1, 2, \ldots\}.
\end{aligned} \tag{2.9}$$

The corresponding $\{G_i^\dagger\}$ are

$$G_1^\dagger = \frac{1}{N_1} b_1^\dagger b_2^\dagger \qquad G_2^\dagger = \frac{1}{N_2} b_1^\dagger b_2^\dagger \qquad G_3^\dagger = \frac{1}{2}(G_1^\dagger + G_2^\dagger). \tag{2.10}$$

The eigenstates of $b_1 b_2$ are, with $m > 0$,

$$\begin{aligned}
&|\alpha\rangle_{m1} \sim \exp\left(\frac{\alpha}{N_1} b_1^\dagger b_2^\dagger\right)|m, 0\rangle \\
&|\alpha\rangle_{m2} \sim \exp\left(\frac{\alpha}{N_2} b_1^\dagger b_2^\dagger\right)|0, m\rangle \\
&|\alpha\rangle_{03} \sim \exp\left\{\frac{\alpha}{2}\left(\frac{1}{N_1} + \frac{1}{N_2}\right) b_1^\dagger b_2^\dagger\right\}|0, 0\rangle.
\end{aligned} \tag{2.11}$$

The eigenstates of $\{G_i\}$ are, respectively, with $m > 0$,

$$\begin{aligned}
&|\alpha\rangle_{m1}' \sim \exp(\alpha b_1^\dagger b_2^\dagger)|m, 0\rangle \\
&|\alpha\rangle_{m2}' \sim \exp(\alpha b_1^\dagger b_2^\dagger)|0, m\rangle \\
&|\alpha\rangle_{03}' \sim \exp(\alpha b_1^\dagger b_2^\dagger)|0, 0\rangle.
\end{aligned} \tag{2.12}$$

The squeezed vacuum, Yuen, pair coherent, Caves–Schumaker and other non-classical states can be identified among the states constructed above (see [35] for details).

It is clear, from the given examples, that the above procedure can be generalized to obtain the eigenstates of $F$ corresponding to three or more bosons. This method is also applicable for constructing the eigenstates of linear combinations of $F$ and $F^\dagger$, such as the squeezed coherent states. One can also use this method to get the thermal counterparts of the various states like the squeezed states, pair coherent states etc (see [35] for details).

## 3. Analogue of the single-photon coherent state for a GBO

For any GBO (1.1) characterized by a $\phi(N)$ with $\phi(N) > 0$, $\forall\, n \geqslant 1$, one can write

$$a = \sqrt{\frac{\phi(N+1)}{N+1}}\,b \qquad a^\dagger = b^\dagger\sqrt{\frac{\phi(N+1)}{N+1}} \qquad N = b^\dagger b \tag{3.1}$$

in terms of the bosonic operators $(b, b^\dagger)$. This follows from the Fock representation of algebra (1.1), in which one can take, with a unique $|0\rangle$ and $\phi(n) > 0\ \forall\, n > 0$,

$$\begin{aligned} &a|0\rangle = 0 \qquad N|n\rangle = n|n\rangle \qquad n = 0, 1, 2, \ldots \\ &a|n\rangle = \sqrt{\phi(n)}|n-1\rangle \qquad a^\dagger|n-1\rangle = \sqrt{\phi(n)}|n\rangle \qquad n = 1, 2, \ldots \\ &|n\rangle = \frac{1}{\sqrt{\phi(n)!}}a^{\dagger n}|0\rangle \end{aligned} \tag{3.2}$$

where $\phi(n)! = \phi(n)\phi(n-1)\ldots\phi(2)\phi(1)$ and $\phi(0)! = 1$. Hence, in the bosonic Fock space, the operator pairs $(a, A^\dagger)$ and $(A, a^\dagger)$, defined by (1.2) and (1.3), have the realization

$$\begin{aligned} &a = TbT^{-1} \qquad A^\dagger = Tb^\dagger T^{-1} \qquad A = T^{-1}bT \qquad a^\dagger = T^{-1}b^\dagger T \\ &T|n\rangle = \sqrt{\frac{n!}{\phi(n)!}}|n\rangle \qquad T^{-1}|n\rangle = \sqrt{\frac{\phi(n)!}{n!}}|n\rangle \qquad n = 0, 1, 2, \ldots \\ &A^\dagger a = a^\dagger A = b^\dagger b = N. \end{aligned} \tag{3.3}$$

This similarity relation between $(b, b^\dagger)$, $(a, A^\dagger)$ and $(A, a^\dagger)$ in the bosonic Fock space implies that in any algebraic relation involving $(b, b^\dagger)$, valid in the bosonic Fock space or a subspace thereof, one can replace $b$ and $b^\dagger$ by $a$ and $A^\dagger$ (or $A$ and $a^\dagger$), respectively, and the resulting relation will be valid in the corresponding GBO Fock spaces. For example, the relation $[b, \exp(\alpha b^\dagger)] = \alpha\exp(\alpha b^\dagger)$, valid in the entire bosonic Fock space, can be translated into the relations $[a, \exp(\alpha A^\dagger)] = \alpha\exp(\alpha A^\dagger)$ and $[A, \exp(\alpha a^\dagger)] = \alpha\exp(\alpha a^\dagger)$, valid in the entire GBO Fock space. Hence, the annihilation-operator eigenstates for the GBO [10, 11, 22] are seen to be given by

$$|\alpha\rangle \sim \exp(\alpha A^\dagger)|0\rangle = \exp\left(\alpha a^\dagger\frac{N+1}{\phi(N+1)}\right)|0\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{\phi(n)!}}|n\rangle \tag{3.4}$$

such that $a|\alpha\rangle = \alpha|\alpha\rangle$. Similarly, one can also define for the GBO an eigenstate of $A$: the state

$$|\alpha\rangle' \sim \exp(\alpha a^\dagger)|0\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\sqrt{\phi(n)!}|n\rangle \tag{3.5}$$

is such that $A|\alpha\rangle' = \alpha|\alpha\rangle'$. The completeness relation for the coherent states in this case becomes

$$\frac{1}{\pi}\int d^2\alpha\,|\alpha\rangle\langle\alpha|' = \frac{1}{\pi}\int d^2\alpha\,|\alpha\rangle'\langle\alpha| = 1. \tag{3.6}$$

Proof of relation (3.6) follows by referring it to the boson realization and using the similarity transformation (3.3). For the usual boson, the two states $|\alpha\rangle$ and $|\alpha\rangle'$ coincide. It may also be noted that if the $|n\rangle$'s in (3.4) and (3.5) are interpreted as the usual bosonic number states then $|\alpha\rangle$ and $|\alpha\rangle'$ are, respectively, the eigenstates of the bosonic operators

$$F = \sqrt{\frac{\phi(N+1)}{(N+1)}}b$$

and

$$G = \sqrt{\frac{(N+1)}{\phi(N+1)}}b$$

satisfying the relations $[F, G^\dagger] = 1$ and $[G, F^\dagger] = 1$.

The coherent state for a GBO is usually defined [2, 26–34] in terms of a generalized exponential, which, in general, can be defined by

$$\exp_\phi(x) = \sum_{n=0}^{\infty} \frac{x^n}{\phi(n)!}. \tag{3.7}$$

It is such that

$$D_\phi \exp_\phi(\alpha x) = \alpha \exp_\phi(\alpha x) \qquad \text{with } D_\phi f(x) = \frac{1}{x}\phi\left(x\frac{\partial}{\partial x}\right) f(x) \tag{3.8}$$

where $D_\phi$ is the generalized derivative, or the $\phi$-derivative, operator. Then, the eigenstate of $a$ is given by

$$|\alpha\rangle \sim \exp_\phi(\alpha a^\dagger)|0\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n a^{\dagger n}}{\phi(n)!}|0\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{\phi(n)!}}|n\rangle. \tag{3.9}$$

Comparison of (3.4) and (3.9) shows the equivalence of the two definitions of the eigenstates of $a$. Similarly, the state $|\alpha\rangle'$ defined above as the eigenstate of $A$ can be written as

$$|\alpha\rangle' \sim \exp_\Phi(\alpha A^\dagger)|0\rangle \qquad \text{with } \Phi(N) = A^\dagger A = \frac{N^2}{\phi(N)} \tag{3.10}$$

with the pair $(A, A^\dagger)$ corresponding to a GBO. The GBOs corresponding to $(a, a^\dagger)$ and $(A, A^\dagger)$ may be regarded as a pair of 'mutually-conjugate' GBOs with their respective characteristic functions $\phi(N)$ and $\Phi(N)$ satisfying the relation $\phi\Phi = N^2$; the usual boson is a 'self-conjugate oscillator'!

Some examples of GBOs are given in table 1 with

$$[X]_{p,q} = \left(\frac{q^X - p^{-X}}{q - p^{-1}}\right) \qquad [X]_q = [X]_{q,q}$$

$$\gamma(g, N) = N + \tfrac{1}{2}(g-1)(1-(-1)^N)$$

$$\wp(g, N) = \gamma(g, N+1) - \gamma(g, N) = 1 + (g-1)(-1)^N.$$

Table 1. Generalized bosonic algebras.

| | $(a, a^\dagger, N)$-relation | $\phi(N)$ | Reference(s) |
|---|---|---|---|
| 1 | $aa^\dagger - qa^\dagger a = q^{-N}$ | $[N]_q$ | [1–4] |
| 2 | $aa^\dagger - qa^\dagger a = 1$ | $[N]_{1,q}$ | [5–11] |
| 3 | $aa^\dagger - qa^\dagger a = q^N$ | $[N]_{q^{-1},q}$ | [18, 20] |
| 4 | $aa^\dagger + qa^\dagger a = q^{-N}$ | $[N]_{q,-q}$ | [12, 13] |
| 5 | $aa^\dagger - qa^\dagger a = p^{-N}$ | $[N]_{p,q}$ | [14–16] |
| 6 | $aa^\dagger - a^\dagger a = \wp(g, N)$ | $\gamma(g, N)$ | [22] |
| 7 | $aa^\dagger - q^{\wp(g,N)} a^\dagger a = [\wp(g, N)]_{1,q}$ | $[\gamma(g, N)]_{1,q}$ | [23,24] |
| 8 | $aa^\dagger - qa^\dagger a = \wp(g, N) q^{-N}$ | $[N]_q + (g-1)[N]_{-q,q}$ | [21] |
| 9 | $aa^\dagger - q^{\wp(g,N)} a^\dagger a = p^{-\gamma(g,N)}[\wp(g, N)]_{p,q}$ | $[\gamma(g, N)]_{p,q}$ | [25] |
| 10 | $aa^\dagger - a^\dagger a = [2N+1]_q$ | $[N]_q^2$ | |

There are several other GBO algebras in the literature (see, e.g., [17–19]). It may be noted that algebra (6) in table 1 is the same as the algebra of a single-mode paraboson of order $g$, usually presented in terms of a triple-commutation relation. It is interesting that this algebra (6) has been recently recognized as being associated with the two-particle Calogero model (see [21] for details). The algebras (7–9) correspond to deformed parabosons.

As already mentioned in the introduction, the commutation relation connecting $a$, $a^\dagger$ and $N$ depends on the recursion relation for $\phi(N)$. In general, this commutation relation can be written in the form

$$aa^\dagger - \xi(N) a^\dagger a = G(N) \tag{3.11}$$

where $\xi(N)$ and $G(N)$ are functions of $N$ and the deformation parameter(s). In [18], several GBOs have been considered in a unified way, taking $\xi(N)$ to be independent of $N$. However, to include GBOs like the $q$-parabose oscillators (see (7 and 9) in the above list) in such a unified picture, one has to consider a relation of the type (3.11). The relation connecting $\xi(N)$, $G(N)$ and $\phi(N)$ is, with $\phi(0) = 0$,

$$\phi(n) = \xi(n-1)! \sum_{k=0}^{n-1} \frac{G(k)}{\xi(k)!} \qquad \text{for } n \geqslant 1. \tag{3.12}$$

Let us now consider the construction of the coherent states of the GBO (10) in the above list, as an example of the formalism detailed above. Equation (3.4) readily gives the result. The eigenstate of $a$ in this case is

$$|\alpha\rangle \sim \sum_{n=0}^{\infty} \frac{\alpha^n}{[n]_q!} |n\rangle. \tag{3.13}$$

From (3.5), it follows that

$$|\alpha\rangle' \sim \sum_{n=0}^{\infty} \frac{\alpha^n [n]_q!}{n!} |n\rangle \tag{3.14}$$

is the eigenstate of the corresponding

$$A = \frac{(N+1)}{[N+1]_q^2} a.$$

The Bargmann–Fock realization of the GBO-algebra is obtained by the map

$$a^\dagger = z \qquad N = z\frac{\partial}{\partial z} \qquad a = \frac{1}{z}\phi\left(z\frac{\partial}{\partial z}\right) \tag{3.15}$$

in the space of analytic functions of the complex variable $z$. The inner-product which makes $(a, a^\dagger)$ Hermitian conjugates in this realization is

$$(f, g) = \left\{ f\left(\frac{1}{z}\phi\left(z\frac{\partial}{\partial z}\right)\right) g(z) \right\}\bigg|_{z=0} \tag{3.16}$$

and the functions $\{z^n/\sqrt{\phi(n)!}\,|n = 0, 1, 2, \ldots\}$ form a complete orthonormal set with respect to this inner-product. Thus, with the correspondence $|n\rangle \longrightarrow z^n/\sqrt{\phi(n)!}$, in the Bargmann–Fock realization, the coherent state has the representation

$$\psi_\alpha(z) \sim \sum_{n=0}^{\infty} \frac{\alpha^n}{\phi(n)!} z^n = \exp_\phi(\alpha z). \tag{3.17}$$

For the realization of $(A, A^\dagger)$, one has

$$A = \frac{\partial}{\partial z} \qquad A^\dagger = z\frac{\partial}{\partial z}\left\{\phi\left(z\frac{\partial}{\partial z}\right)\right\}^{-1} z. \tag{3.18}$$

The state $|\alpha\rangle'$ is given by

$$\psi'_\alpha(z) \sim \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} z^n = \exp(\alpha z) \tag{3.19}$$

as is obvious from the realization $A = \partial/\partial z$.

For the GBO (10), in the Bargmann–Fock realization, we have

$$\psi_\alpha(z) \sim \sum_{n=0}^{\infty} \frac{\alpha^n}{([n]_q!)^2} z^n \tag{3.20}$$

which is a $q$-deformed version of the modified Bessel function $I_0(2\sqrt{\alpha z})$. By definition, $\psi_\alpha(z)$ in (3.20) satisfies the eigenvalue equation

$$\frac{1}{z}\left[z\frac{\partial}{\partial z}\right]_q^2 \psi_\alpha(z) = \frac{\psi_\alpha(q^2 z) - 2\psi_\alpha(z) + \psi_\alpha(q^{-2}z)}{(q - q^{-1})^2 z} = \alpha\psi_\alpha(z). \tag{3.21}$$

In the limit $q \longrightarrow 1$, the GBO-algebra (10) becomes the classical $su(1,1)$ algebra ($a \longrightarrow K_-$, $a^\dagger \longrightarrow K_+$, $N \longrightarrow K_0 - \frac{1}{2}$) and, hence, the states

$$|\alpha\rangle \sim \sum_{n=0}^{\infty} \frac{\alpha^n a^{\dagger n}}{([n]_q!)^2} |0\rangle \tag{3.22}$$

may be recognized as a $q$-generalization of the Barut–Girardello $su(1,1)$ coherent states [36].

## 4. Analogues of the multiphoton coherent states for a GBO

Analogues of the multiphoton coherent states for a GBO can be obtained by the same procedure as above, namely, replacing $(b, b^{\dagger}, N)$'s by $(a, A^{\dagger}, N)$'s and $(A, a^{\dagger}, N)$'s. As the first example, let us consider the construction of the eigenstates of $\mathcal{F} = a^2$. Then, following the above argument, we see, from (2.5), that

$$\mathcal{G}_1^{\dagger} = \frac{1}{2(N-1)} A^{\dagger 2} = \frac{N}{2\phi(N)\phi(N-1)} a^{\dagger 2} \tag{4.1}$$

and

$$\mathcal{G}_2^{\dagger} = \frac{1}{2N} A^{\dagger 2} = \frac{N-1}{2\phi(N)\phi(N-1)} a^{\dagger 2} \tag{4.2}$$

are such that

$$[\mathcal{F}, \mathcal{G}_1^{\dagger}]|2n\rangle = |2n\rangle \qquad [\mathcal{F}, \mathcal{G}_2^{\dagger}]|2n+1\rangle = |2n+1\rangle \qquad n = 0, 1, 2, \ldots. \tag{4.3}$$

Hence, it follows that

$$|\alpha\rangle_1 \sim \exp\left(\frac{\alpha N}{2\phi(N)\phi(N-1)} a^{\dagger 2}\right)|0\rangle \qquad |\alpha\rangle_2 \sim \exp\left(\frac{\alpha(N-1)}{2\phi(N)\phi(N-1)} a^{\dagger 2}\right)|1\rangle \tag{4.4}$$

are two distinct eigenstates of $a^2$ corresponding to the eigenvalue $\alpha$. The eigenstates of

$$\mathcal{G}_1 = \frac{N+2}{2\phi(N+1)\phi(N+2)} a^2$$

and

$$\mathcal{G}_2 = \frac{N+1}{2\phi(N+1)\phi(N+2)} a^2$$

are, respectively,

$$|\alpha\rangle_1' \sim \exp(\alpha a^{\dagger 2})|0\rangle \qquad |\alpha\rangle_2' \sim \exp(\alpha a^{\dagger 2})|1\rangle. \tag{4.5}$$

If we take $\mathcal{F} = A^2$, then, by the same arguments, one has that

$$|\alpha\rangle_1 \sim \exp\left(\frac{\alpha}{2(N-1)} a^{\dagger 2}\right)|0\rangle \qquad |\alpha\rangle_2 \sim \exp\left(\frac{\alpha}{2N} a^{\dagger 2}\right)|1\rangle \tag{4.6}$$

are two distinct eigenstates of $A^2$ with eigenvalue $\alpha$ and where

$$|\alpha\rangle_1' \sim \exp\left(\frac{\alpha N(N-1)}{\phi(N)\phi(N-1)} a^{\dagger 2}\right)|0\rangle \qquad |\alpha\rangle_2' \sim \exp\left(\frac{\alpha N(N-1)}{\phi(N)\phi(N-1)} a^{\dagger 2}\right)|1\rangle$$

are the eigenstates of $\frac{1}{2(N+1)}a^2$ and $\frac{1}{2(N+2)}a^2$, respectively, with eigenvalue $\alpha$.

Extending the above procedure to the GBO analogues of the other two-photon coherent states, we get the results summarized in the table 2 with $(a_1, a_1^{\dagger})$ and $(a_2, a_2^{\dagger})$ corresponding to two commuting GBOs.

The GBO analogues of the various multiphoton coherent states like the squeezed vacuum, Yuen, pair coherent, Caves–Schumaker and other non-classical states can be identified among the states constructed above. In table 2, one can replace $(a_1, a_2)$ in $\mathcal{F}$ with $(A_1, A_2)$, respectively, and *vice versa*, and construct the corresponding eigenstates by replacing $(A_1^{\dagger}, A_2^{\dagger})$ by $(a_1^{\dagger}, a_2^{\dagger})$, respectively, and *vice versa*. As already mentioned, using the bosonic representation of $(a, a^{\dagger})$, one can also interpret the above GBO states as non-classical states of the usual bosonic field.

Table 2. Deformed multimode coherent states.

| $\mathcal{F}$ | eigenstates | |
|---|---|---|
| $a_1 a_2$ | $\sim \exp\left(\frac{\alpha}{N_1} A_1^\dagger A_2^\dagger\right) \vert m, 0\rangle$ | $m > 0$ |
| | $\sim \exp\left(\frac{\alpha}{N_2} A_1^\dagger A_2^\dagger\right) \vert 0, m\rangle$ | $m > 0$ |
| | $\sim \exp\left(\frac{\alpha}{2}\left\{\frac{1}{N_1} + \frac{1}{N_2}\right\} A_1^\dagger A_2^\dagger\right) \vert 0, 0\rangle$ | |
| $\frac{1}{N_1+1} A_1 A_2$ | $\sim \exp(\alpha a_1^\dagger a_2^\dagger) \vert m, 0\rangle$ | $m > 0$ |
| $\frac{1}{N_2+1} A_1 A_2$ | $\sim \exp(\alpha a_1^\dagger a_2^\dagger) \vert 0, m\rangle$ | $m > 0$ |
| $\frac{1}{2}\left(\frac{1}{N+1} + \frac{1}{N+2}\right) A_1 A_2$ | $\sim \exp(\alpha a_1^\dagger a_2^\dagger) \vert 0, 0\rangle$ | |

## 5. Conclusion

To conclude, we have presented a general unified formalism for obtaining the analogues of the single-photon and the various multiphoton coherent states in the case of a GBO (for a different approach to multimode $q$-coherent states see [37]). To this end, we have used the similarity relation that exists between the annihilation (creation) operators of the GBO and the usual bosonic oscillator to apply the technique developed recently [35] for obtaining the large class of bosonic coherent states known in the literature in a unified way.

## Acknowledgments

One of us (RJ) is thankful to Professor G S Agarwal for fruitful discussions and kind hospitality during the Workshop on 'Coherent States: New Developments and Perspectives' (School of Physics, University of Hyderabad, Hyderabad, October 29–31, 1993) when part of this work was done.

*Note added.* For deformed parabosonic algebras see also Celeghini E, Palev T D and Tarlini M 1991 *Mod. Phys. Lett. B* **5** 187 and Palev T D 1993 *J. Phys. A: Math. Gen.* **26** L1111 and references therein.

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